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Parameterized variational inequalities

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Abstract We prove that under suitable conditions, the solution set of a variational inequality, governed by perturbed monotone operators depending on a parameter, has a continuous selection. In the nonparametric case this can be considered as a variational principle for variational inequalities, an analogue of the Borwein–Preiss variational principle. An applications of this result is given.

Keywords Variational inequality · Monotone operators · Variational principle

1 Preliminaries

It is well known that the variational inequality (A, Y), defined by

$$\text{find } y_0 \in Y \quad \text{such that } \langle A(y_0), y_0 - y\rangle \le 0 \qquad \forall y \in Y \tag{1}$$

has a solution, if $A : Y \to E^*$ is a monotone operator, continuous on the finite dimensional subsets of Y and Y is a weakly compact convex subset of a Banach space E with dual E^* (see [7, Theorem 1.4, Ch. III]). If Y is not weakly compact, then (1) may have no solutions. For example, let Y be a convex, closed and bounded subset of a non-reflexive Banach space with Frechét differentiable norm off 0, $0 \notin Y$ and Y has no nearest point to 0. Such Y can be easily constructed by the James theorem (see for instance [4]). Then the variational inequality $(\nabla\|.\|, Y)$ has no solutions (here and further $\nabla\varphi$ denotes the Frechet derivative of the function φ). Nevertheless, there are arbitrary small monotone perturbations of the motonote operator

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$\nabla\|.\|$ such that for the perturbed operators the variational inequality (1) has a solution. This fact follows easily by the Borwein–Preiss variational principle [2].

One of the aim of this paper is to prove such a result for a variational inequality (A, Y), where Y is a closed and convex subset of a Fréchet smooth Banach E space and $A : Y \to E^*$ is a monotone operator, continuous on the finite dimensional subsets of Y, satisfying the following condition:

$$\inf_{z\in Y} \sup_{y\in Y} \langle A(y), z - y\rangle = 0. \tag{2}$$

Namely, we prove that for every $\varepsilon > 0$ there exists a monotone operator Γ of norm 1 such that the variational inequality $(A + \varepsilon\Gamma, Y)$ has a solution.

Such a result can be regarded as an analogue of the Borwein–Preiss variational principle for monotone operators.

Another aim of this paper is to give a parametric version of the above result. This is of the same spirit, as the parametric versions of the Ekeland variational principle and of the Borwein–Preiss variational principle, considered in [5,6]. As application here an analogue of the notion 'Nash equilibrium' is considered for monotone operators and an existence result for the perturbed problem is proved (see Theorem 3.1).

2 Main result

Let $(E, \|.\|)$ be a Fréchet smooth Banach space, X be a paracompact topological space, Y be a closed, convex and non-empty subset of E.

Theorem 2.1 *Suppose that $A : X \times Y \to E^*$ is a mapping such that $A(x, .) : E \to E^*$ is a monotone operator, continuous on the finite dimensional subsets of Y for every $x \in X$, and the family of mappings $\{A(., y) : y \in Y\}$ is equi-continuous. Assume that*

$$\inf_{z\in Y} \sup_{y\in Y} \langle A(x, y), z - y\rangle = 0 \quad \forall x \in X. \tag{3}$$

Let $\varepsilon > 0$ and $y_0 : X \to Y$ be a continuous maping such that

$$\langle A(x, y), y_0(x) - y\rangle \le \varepsilon \quad \forall x \in X, \quad \forall y \in Y.$$

Then for every $\lambda > 0$, $p > 1$ there exists a continuous mapping $v : X \to Y$ such that

$$\left\langle \left(A + \frac{\varepsilon}{\lambda^p}\Gamma\right)(x, v(x)), v(x) - y\right\rangle \le 0 \quad \forall x \in X, \quad \forall y \in Y, \tag{4}$$

$$\|v(x) - y_0(x)\| < \lambda \quad \forall x \in X, \tag{5}$$

where

$$\Gamma(x, y) = \nabla\left(\sum_{n=0}^{\infty} \mu_n \|y - y_n(x)\|^p\right), \qquad \sum_{n=0}^{\infty} \mu_n = 1, \tag{6}$$

$y_n : X \to Y$ are continuous mappings, converging uniformly to v.

Proof Put

$$K(q,\delta)=\left[\frac{q^2+\frac{\delta}{1-\delta}}{p\delta(1-q)(1-\delta)^{p-1}}\right]^{\frac{1}{p}}\frac{1}{1-q^{\frac{1}{p}}}$$

and since

$$\lim_{\delta\to 0_+}\lim_{q\to 0+}(K(q,\delta))=p^{-\frac{1}{p}},$$

there exists $q,\delta\in(0,1)$ such that $K(q,\delta)<1$. Let $\varepsilon_n=q^{2n}\varepsilon$, $\mu_n=q^n(1-q)$.

Define inductively

$$A_{n+1}(x,y)=A_n(x,y)+\frac{\varepsilon}{\lambda^p}\mu_n\nabla\|y-y_n(x)\|^p,\quad A_0=A, \tag{7}$$

where $y_n:X\to Y$ is a continuous mapping such that

$$\sup_{y\in Y}\langle A_n(x,y),y_n(x)-y\rangle\le\varepsilon_n. \tag{8}$$

We shall prove by induction that such a definition is possible. For $n=0$ (8) is satisfied by assumption. Assume that (8) is fulfilled for some n.

Claim *The mulivalued mapping* $F:X\to 2^Y$ *defined by*

$$F(x)=\{z\in Y:\sup_{y\in Y}\langle A_n(x,y),z-y\rangle\le\varepsilon_n\}$$

is lower semicontinuous with convex and closed images.

Proof of the Claim The convexity and the closedness of the images of F is evident.

Let $x_0\in X$. **Case 1** Let $z_0\in Y$ be such that

$$\sup_{y\in Y}\langle A_n(x_0,y),z_0-y\rangle<\varepsilon_n$$

Since the norm is Frechét differentiable, it is a routine matter to prove that for sufficiently small $\delta>0$

$$\sup_{y\in Y}\langle A_n(x,y),z_0-y\rangle<\varepsilon_n\qquad\forall x\in B(x_0;\delta),$$

so

$$z_0\in F(x)\qquad\forall x\in X. \tag{9}$$

Case 2 Let now $z_0\in Y$ be such that

$$\sup_{y\in Y}\langle A_n(x_0,y),z_0-y\rangle=\varepsilon_n$$

By (3), there exists $z_1\in Y$ such that

$$\sup_{y\in Y}\langle A_n(x_0,y),z_1-y\rangle<\varepsilon_n$$

Then

$$\sup_{y\in Y}\langle A_n(x_0, y), \lambda z_1 + (1-\lambda)z_0 - y\rangle$$
$$\leq \lambda \sup_{y\in Y}\langle A_n(x_0, y), z_1 - y\rangle + (1-\lambda)\sup_{y\in Y}\langle A_n(x_0, y), z_0 - y\rangle < \varepsilon_n \quad (10)$$

for every $\lambda > 0$. Now by Case 1, the proof of the Claim is completed.

By the Claim and by the Michael's selection theorem we can find a continuous selection y_n of F_n, which proves the correctness of the definition.

Putting $y = \delta y_n(x) + (1-\delta)y_{n+1}$ by (7) and (8), for $n+1$, we obtain:

$$\begin{aligned}\frac{p\varepsilon}{\lambda^p}\mu_n(1-\delta)^{p-1}\delta\|y_{n+1}(x) - y_n(x)\|^p &\leq \varepsilon_{n+1} - \delta\langle A_n(x, y), y_{n+1}(x) - y_n(x)\rangle\\ &= \varepsilon_{n+1} + \frac{\delta}{1-\delta}\langle A_n(x, y), y_n(x) - y\rangle\\ &\leq \varepsilon_{n+1} + \frac{\delta}{1-\delta}\varepsilon_n. \end{aligned} \quad (11)$$

Hence

$$\begin{aligned}\|y_{n+1}(x) - y_n(x)\| &\leq \lambda\left[\frac{e_{n+1} + \frac{\delta}{1-\delta}\varepsilon_n}{p\varepsilon(1-\delta)^{p-1}\delta\mu_n}\right]^{\frac{1}{p}}\\ &= \lambda K(q, \delta)q^{\frac{n}{p}}(1 - q^{\frac{1}{p}}),\end{aligned} \quad (12)$$

therefore

$$\|y_m(x) - y_n(x)\| < \lambda K(q, \delta)q^{\frac{n}{p}}(1 - q^{(m-n)p}) \quad \forall x \in X, \quad (13)$$

which shows that $\{y_n(x)\}_{n=1}^{\infty}$ is a fundamental sequence. Let $v(x)$ be its limit. It is easy to see that v is continuous. When $n = 0$ and m tends to infinity, by (13) we obtain (5).

We shall prove that

$$\langle\tilde{A}(x, y), v(x) - y\rangle \leq 0 \quad \forall x \in X,\; y \in Y, \quad (14)$$

where $\tilde{A} = A + \frac{\varepsilon}{\lambda^p}\Gamma$.

Assume the contrary: for some $x' \in X$ and $y' \in Y$ we have

$$\langle\tilde{A}(x', y'), v(x') - y'\rangle > 0.$$

Take $0 < \varepsilon_0 < \langle\tilde{A}(x', y'), v(x') - y'\rangle$ and an integer n such that

$$\|v(x') - y_n(x')\| < \frac{\varepsilon_0}{3\|\tilde{A}(y')\|}, \quad \varepsilon_n < \frac{\varepsilon_0}{3} \quad \text{and} \quad \frac{p\varepsilon}{\lambda^p}\sum_{k=n}^{\infty}\mu_n < \frac{\varepsilon_0}{3(diamY)^p}.$$

Then we have (using (7) and (8)):

$$
\begin{aligned}
&\langle \tilde{A}(x', y'), v(x') - y' \rangle \\
&= \langle \tilde{A}(x', y'), v(x') - y_n(x') \rangle + \langle A_n(x', y'), y_n(x') - y' \rangle \qquad (15)\\
&\quad + \langle (\tilde{A} - A_n)(x', y'), y_n(x') - y' \rangle \\
&\leq \|\tilde{A}(x', y')\| \|v(x') - y_n(x')\| + \varepsilon_n \qquad (16)\\
&\quad + \frac{p\varepsilon}{\lambda^p} \sum_{k=n}^{\infty} \mu_k \|y_k(x') - y'\|^{p-1} \langle \nabla \|y_k(x') - y'\|, y_n(x') - y' \rangle \\
&< \frac{2\varepsilon_0}{3} + \frac{p\varepsilon}{\lambda^p} (diamY)^p \sum_{k=n}^{\infty} \mu_k < \varepsilon_0,
\end{aligned}
$$

a contradiction, and (14) is proved. Now we apply Minty's lemma (see [7, Lemma 1.5, Ch. III], and we obtain (4).

Remark 2.2 The condition (3) is satisfied, if Y is, in addition, weakly compact. This follows by the clasical theorem for existence of solutions of variational inequalities [7].

Remark 2.3 When X consists of one element (i.e. when F does not depend on x), then Theorem 2.1 can be considered as an analogue of the Borwein–Preiss variational principle for monotone operators.

Open question. *For which non-weakly compact closed convex sets Y, (2) is satisfied?*

3 Application to Nash equilibrium problems for variational inequalities

The next theorem gives existence of Nash equilibrium for finite numbers of perturbed variational inequalities. It is an analogue of the classical theorem of Nash for convex functions (see [1, Theorem 13, Ch.6]). The difference is that here one of the sets is allowed to be non-compact, and the result is obtained after (arbitrarily small) perturbation of the monotone operators defining the variational inequalities.

Theorem 3.1 *Let $E_i, i = 1, \ldots, n$ be Fréchet smooth Banach spaces, X_i be convex and compact subset of E_i for $i = 2, \ldots, n$, X_1 be convex, closed and bounded subset of E_1. Denote $X = X_1 \times \cdots \times X_n$, $x = (x_1, \ldots, x_n)$, $x_{\hat{i}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $X_{\hat{i}} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$, $\forall i = 1, \ldots, n$. Let the mappings $A_i : X \to E_i^*$ be monotone with respect to the variable $x_i \in E_i$ and the family $\{A_i(\ldots, x_i, \ldots) : x_i \in X_i\}$ be equi-continuous on $X_{\hat{i}}$ for every $i = 1, \ldots, n$. Then for every $\varepsilon > 0$ there exist monotone operators Γ_i of norm less than ε, there exists $x^0 \in X$ such that:*

$$\langle A_i(x^0) + \Gamma_i(x_i^0), x_i^0 - y \rangle \leq 0 \quad \forall y \in X_i, \ \forall i = 1, \ldots, n.$$

Proof From Theorem 2.1 for every $i = 1, \ldots, n$ there exists a continuous mapping $v_i : X_{\hat{i}} \to X_i$ and a mappings $\Delta_i : X \to E_i^*$ of norm less than ε which is monotone with respect to the variable $x_i \in E_i$ such that

$$\langle (A_i + \Delta_i)(x_1, \ldots, v_i(x_{\hat{i}}), \ldots, x_n), v_i(x_{\hat{i}}) - y \rangle \leq 0, \qquad \forall y \in X_i.$$

The composition mapping

$$X_2 \times \cdots \times X_n \ni x_{\hat{1}} \to (v_2(\varphi_2(x_{\hat{1}})), \ldots, v_n(\varphi_n(x_{\hat{1}}))) \in X_2 \times \cdots \times X_n,$$

where $\varphi_i(x_{\hat{1}}) = (v_1(x_{\hat{1}}), x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $i = 2, \ldots, n$ is a continuous mapping from the compact convex set $X_2 \times \cdots \times X_n$ to itself and from Shauder's fixed point theorem it has a fixed point $\bar{x}_{\hat{1}} = (\bar{x}_2, \ldots, \bar{x}_n)$. If we put $\bar{x}_1 = y_1(\bar{x}_{\hat{1}})$ and $\Gamma_i(x_i) := \Delta_i(\bar{x}_1, \ldots, x_i, \ldots, \bar{x}_n)$, then $\bar{x}_i = y_i(\bar{x}_{\hat{i}})$ for every $i = 2, \ldots, n$, and the proof is completed. □

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